

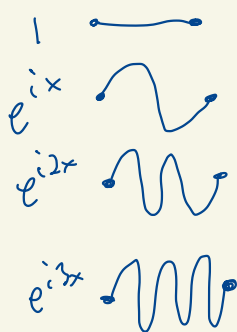
Monday, November 10, Lecture 30: The Fourier Transform on $\mathbb{R}$

Fourier Transform on the Circle

So far, we have studied

$$f \in L^2([-\pi, \pi]), \text{ periodic boundary conditions} \xrightleftharpoons[U^\dagger]{U} \hat{f} \in \ell^2(\mathbb{Z})$$

onbs: $\{e^{ikx} : k \in \mathbb{Z}\}$



$\{\delta_k : k \in \mathbb{Z}\}$

$$\delta_0 = (\dots, 0, 1, 0, 0, 0, \dots)$$

$$\delta_1 = (\dots, 0, 0, 1, 0, 0, \dots)$$

$$\delta_2 = (\dots, 0, 0, 0, 1, 0, \dots)$$

$$\delta_3 = (\dots, 0, 0, 0, 0, 1, \dots)$$

Parseval: This map is unitary,

$$\begin{array}{ccc} \langle f, g \rangle & = & \langle \hat{f}, \hat{g} \rangle \\ \uparrow & & \uparrow \\ L^2([-\pi, \pi]) & & \ell^2(\mathbb{Z}) \end{array}$$

• Define $L^2(\mathbb{R}) = \left\{ f: \mathbb{R} \rightarrow \mathbb{C} \text{ s.t. } \|f\| < \infty \right\}$,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx$$

To understand, let's think about $L^2([-L, L])$.

Fundamental harmonic

$$e^{ix}$$



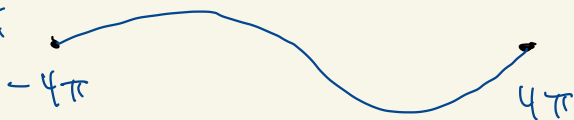
$$k = \dots, -1, 0, 1, \dots$$

$$e^{i\frac{x}{2}}$$



$$k = \dots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \dots$$

$$e^{i\frac{x}{4}}$$



$$k = \dots, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \dots$$

as $L \rightarrow \infty$, supports longer and longer waves

$\Rightarrow$ frequency $k \rightarrow 0$.

• Suggests we can understand f by decomposing into waves e^{ikx} with frequency $k \in \mathbb{R}$!

Warning: waves e^{ikx} are not square integrable on $\mathbb{R}$: e.g. if $k=0$,

$$\int_{-\infty}^{\infty} 1 dx = \infty$$

So $e^{ikx} \notin L^2(\mathbb{R})$, although we will still be able to work with them.

But look: if $f \in L^2(\mathbb{R})$, the Fourier coefficient

$$\hat{f}(k) = \langle f, e^{ikx} \rangle = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

still makes sense for $k \in \mathbb{R}$, since

$$\int_{-\infty}^{\infty} |f(x) e^{ikx}|^2 dx \leq \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$< \infty$$

Def: Fourier Transform, Inverse Fourier Transform
 $f: \mathbb{R} \rightarrow \mathbb{C}, \hat{f}: \mathbb{R} \rightarrow \mathbb{C}$

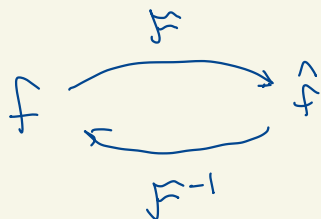
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

Inverse Fourier Transform

So: the Fourier transform F is an invertible linear map, which means



$$\text{or, } \hat{\hat{f}} = f$$

If we know that our functions $f \in L^2(\mathbb{R})$, we get that the Fourier Transform is unitary.

Parseval's Thm, $L^2(\mathbb{R})$

The map $F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is unitary,
 $f \mapsto \hat{f}$

$$\int_{-\infty}^{\infty} \overline{f(x)} g(x) dx = \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle = \int_{-\infty}^{\infty} \overline{\hat{f}(k)} \hat{g}(k) dk$$

But we can still take Fourier transforms of other functions, like e^{ikx} .

Ex:

Compute the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ik} - e^{ik}}{(-ik)} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin k}{k} \end{aligned}$$

This is a commonly encountered function, called the sinc function

$$\text{sinc } k = \frac{\sin k}{k}$$



By Fourier inversion,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(k) e^{ikx} dk = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin k}{k} e^{ikx} dk$$

Notice: if $x=0$,

$$1 = f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin k}{k} dk \Leftrightarrow \pi = \int_{-\infty}^{\infty} \frac{\sin k}{k} dk$$

This was the tricky contour integral from a while ago!

Many features of Fourier Series still hold.

Prop: (Fourier Transform diagonalizes derivatives)

$$\widehat{-i \frac{d}{dx} f} = k \hat{f}$$

equivalently,

$$\mathcal{F}(-i \frac{d}{dx} f) = k \cdot \hat{f}$$

$$-i \frac{d}{dx} (f) = \mathcal{F}^{-1} \cdot k \cdot \mathcal{F}(f)$$

pf:

Assume $f(x) \rightarrow 0$ as $x \rightarrow \infty$ or $-\infty$.

$$\begin{aligned} \widehat{-i \frac{d}{dx} f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i \frac{d}{dx} f e^{-ikx} dx \\ \text{IBP: } \begin{matrix} f(x) \rightarrow 0 \\ x \rightarrow \pm\infty, \\ |e^{-ikx}| = 1 \end{matrix} &= \frac{1}{\sqrt{2\pi}} \left(-if e^{-ikx} \Big|_{x=-\infty}^{\infty} + i \int_{-\infty}^{\infty} (-ik) f(x) e^{ikx} dx \right) \\ &= k \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= k \hat{f}(k) \quad \square \end{aligned}$$

In fact, this works more generally, even if f doesn't go to 0 as $k \rightarrow \infty$!